# AR($p$): Definition, stability and stationarity

A time series follows a zero-mean autoregressive process of order $p$, of AR($p$), if

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \epsilon_t,$$

with $\epsilon_t \sim^{iid} N(0, v)$, for all $t$.

The AR characteristic polynomial is given by

$$\Phi(u) = 1 - \phi_1 u - \phi_2 u^2 - \ldots - \phi_p u^p,$$

with $u$ complex-valued.

The AR($p$) process is *stable* if $\Phi(u) = 0$ only when $|u| > 1$. In this case the process is also stationary and can be written as

$$y_t = \psi(B)\epsilon_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j},$$

with $\psi_0 = 1$ and $\sum_{j=0}^{\infty} |\psi_j| < \infty$. Here $B$ denotes the backshift operator and so $B^j \epsilon_t = \epsilon_{t-j}$ and

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots + \psi_j B^j + \ldots$$

The AR polynomial can also be written as

$$\Phi(u) = \prod_{j=1}^{p} (1 - \alpha_j u)$$

with $\alpha_j$ the reciprocal roots of the characteristic polynomial. In this case for the process to be stable (and consequently stationary), $|\alpha_j| < 1$ for all $j = 1 : p$.

## AR($p$) : State-space representation

An AR($p$) can also be represented using the following state-space or dynamic linear (DLM) model representation:

$$\begin{aligned} y_t &= \boldsymbol{F}'\boldsymbol{x}_t \\ \boldsymbol{x}_t &= \boldsymbol{G}\boldsymbol{x}_{t-1} + \boldsymbol{\omega}_t, \end{aligned}$$

with $\boldsymbol{x}_t = (y_t, y_{t-1}, \dots, y_{t-p+1})'$, $\boldsymbol{F} = (1, 0, \dots, 0)'$, $\boldsymbol{\omega}_t = (\epsilon_t, 0, \dots, 0)'$. and

$$\boldsymbol{G} = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Using this representation we have that the expected behavior of the process in the future can be exhibited via the forecast function:

$$f_t(h) = E(y_{t+h}|y_{1:t}) = \boldsymbol{F}'\boldsymbol{G}^h\boldsymbol{x}_t, \quad h > 0,$$

for any $t \geq p$. The eigenvalues of the matrix $\boldsymbol{G}$ are the reciprocal roots of the characteristic polynomial. Note that they can be real-valued or complex-valued. Complex-valued eigenvalues/reciprocal roots appear in conjugate pairs.

Assuming the matrix $\boldsymbol{G}$ has $p$ distinct eigenvalues we can decompose $\boldsymbol{G}$ into $\boldsymbol{G} = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^{-1}$, with $\boldsymbol{\Lambda} = diag(\alpha_1, \dots, \alpha_p)$ for a matrix of corresponding eigenvectors $\boldsymbol{E}$. Then, $\boldsymbol{G}^h = \boldsymbol{E}\boldsymbol{\Lambda}^h\boldsymbol{E}^{-1}$ and we have:

$$f_t(h) = \sum_{j=1}^{p} c_{tj}\alpha_j^h.$$

## ACF of AR($p$)

For a general AR($p$), the ACF is given in terms of the homogenous difference equation:

$$\rho(h) - \phi_1\rho(h-1) - \ldots - \phi_p\rho(h-p) = 0, \quad h > 0.$$

Assuming that $\alpha_1, \ldots, \alpha_r$ denote the characteristic reciprocal roots each with multiplicity $m_1, \ldots, m_r$, respectively, with $\sum_{i=1}^r m_i = p$. Then, the general solution is

$$\rho(h) = \alpha_1^h p_1(h) + \ldots + \alpha_r^h p_r(h),$$

with $p_j(h)$ a polynomial of degree $m_j - 1$.

### Example: AR(1)

We already know that for $h \geq 0$, $\rho(h) = \phi^h$. Using the result above we have that

$$\rho(h) = a\phi^h,$$

and so to find $a$ we simply take $\rho(0) = 1 = a\phi^0$ and so $a = 1$.

### Example: AR(2)

Similarly, using the result above in the case of two complex-valued reciprocal roots we have that

$$\rho(h) = a\alpha_1^h + b\alpha_2^h = cr^h \cos(\omega h + d).$$

## PACF of AR($p$)

We can use the Durbin-Levinson recursion to obtain the PACF of an AR($p$) and, using the same representation but subsituting the true autocovariances and autocorrelations by their sampled versions, we can also obtain the sample PACF. It is possible to show that the PACF of an AR($p$) is equal to zero as $h > p$.